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STRUCTURE OF GENETIC ALGEBRAS.*

By R. D. SCHAFER.

I. M. H. Etherington has studied the non-associative algebras which arise in the symbolism of genetics (references [5] through [10]). In these he defines a class of algebras called *train algebras*, and proves in [7] that this class includes algebras called *special train algebras* which are defined by their structure rather than by any type of recurrence equation.

From the algebraic point of view the concept of train algebra appears to be too inclusive, in that an analysis of the structure of train algebras seems feasible only when the rank of the algebra is small. However, from the point of view of genetics the concept of special train algebra is certainly too narrow. For, although the *gametic* algebras for the fundamental types of symmetrical inheritance are special train algebras, the corresponding *zygotic* (*copular*, etc.) algebras are not necessarily special train algebras [7, p. 6, footnote].

We introduce a concept of *genetic algebra* which is intermediate between (commutative) train algebra and special train algebra. The definition is more satisfactory than that of special train algebra on two counts: the structure of the algebra is not postulated, and the *duplicate* of a genetic algebra is a genetic algebra. It follows from this latter fact that our genetic algebras include, not only the fundamental symmetrical gametic algebras, but also the zygotic (copular, etc.) algebras obtained from them by duplication. On the other hand, this new concept is restrictive enough for us to deduce a transparent structure theory for genetic algebras.

It is only fair perhaps to caution the reader that our interest in these algebras is entirely in the algebraic formalism, and that we can give no indication beyond Etherington's own remarks in [5] and [8] of their possible contribution to the study of genetics. Also we use the name "genetic algebra" with some misgivings. Our results are applicable to the algebras arising in genetics where inheritance is symmetrical in the sexes, and we abbreviate "genetic algebra of symmetrical inheritance" to "genetic algebra."

1. Preliminaries. The principal tool of our investigation of genetic algebras is the *transformation algebra* [1, § 2]. Let \mathfrak{A} be a non-associative

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algebra of order n over a field \mathfrak{F} . Then for a fixed element x in \mathfrak{A} the correspondences

$$a \rightarrow ax = aR_x, \quad a \rightarrow xa = aL_x, \quad \text{for all } a \text{ in } \mathfrak{A},$$

are linear transformations on \mathfrak{A} called the *right* and *left multiplications* R_x and L_x respectively. If \mathfrak{M} is a subset of the total matrix algebra $(\mathfrak{F})_n$ of all linear transformations on \mathfrak{A} , the *enveloping algebra* of \mathfrak{M} is the algebra of all polynomials in the transformations in \mathfrak{M} with coefficients in \mathfrak{F} . The enveloping algebra of the set which consists of the identity I in $(\mathfrak{F})_n$, together with the right and left multiplications of \mathfrak{A} , is the *transformation algebra* $T(\mathfrak{A})$ of \mathfrak{A} . Clearly any T in $T(\mathfrak{A})$ may be written in the form

$$(1) \quad T = \alpha I + f(R_{x_1}, L_{x_1}, R_{x_2}, \dots), \quad \alpha \text{ in } \mathfrak{F}, x_i \text{ in } \mathfrak{A}.$$

If \mathfrak{B} is any linear subspace of \mathfrak{A} , the enveloping algebra of the set of right and left multiplications of \mathfrak{A} which correspond to elements in \mathfrak{B} is denoted by \mathfrak{B}^* . That is, T in \mathfrak{B}^* has the form (1) with $\alpha = 0$, x_i in \mathfrak{B} . (It should be noted that the transformations in \mathfrak{B}^* are linear transformations on \mathfrak{A} , although for compactness the notation does not indicate this.)

A *homomorphism* H of an algebra \mathfrak{A} over \mathfrak{F} into an algebra \mathfrak{C} over \mathfrak{F} is a linear mapping of \mathfrak{A} into \mathfrak{C} such that

$$(2) \quad (ax)H = aH \circ xH \quad \text{for all } a, x \text{ in } \mathfrak{A},$$

where \circ denotes multiplication in \mathfrak{C} . The *kernel* of H is the set \mathfrak{B} of all b in \mathfrak{A} such that $bH = 0$; \mathfrak{B} is an ideal of \mathfrak{A} . The homomorphism is *onto* \mathfrak{C} in case, for any c in \mathfrak{C} , there exists some a in \mathfrak{A} such that $c = aH$. The relationships between homomorphisms, ideals, and difference algebras are well-known. In case the homomorphism in question is from \mathfrak{A} into the base field \mathfrak{F} , we use a functional notation:

$$(3) \quad \omega: x \rightarrow \omega(x), \quad x \text{ in } \mathfrak{A}, \omega(x) \text{ in } \mathfrak{F},$$

and (2) becomes

$$(4) \quad \omega(ax) = \omega(a)\omega(x), \quad \text{for all } a, x \text{ in } \mathfrak{A}.$$

We call an algebra \mathfrak{A} *nilpotent* in case there exists an integer t such that every product of t elements in \mathfrak{A} , no matter how associated, is zero. This is what Albert has recently called *strongly nilpotent* [2, p. 528; 3, p. 549]. He has defined a nilpotent algebra in the following way: every sequence a_1, \dots, a_k of k elements of \mathfrak{A} defines a special product $a^{(k)}$ of order k by

either of the formulas $a^{(i)} = a^{(i-1)}a_i$ or $a^{(i)} = a_i a^{(i-1)}$ for $i > 1$; if all special products of order k are zero and some special product of order $k - 1$ is not zero, Albert calls \mathfrak{A} *nilpotent of index k* . Certainly a strongly nilpotent algebra is nilpotent by these definitions. However, an observation of Etherington [7, p. 2] shows the equivalence of the two notions: for if every special product of index k in \mathfrak{A} is zero, then every product of $t = 2^{k-1}$ elements in \mathfrak{A} , no matter how associated, is zero. Thus the concept of a *strongly* nilpotent algebra is redundant.

A necessary and sufficient condition that an ideal \mathfrak{B} of a non-associative algebra \mathfrak{A} be nilpotent is that the associative algebra \mathfrak{B}^* be nilpotent [2, Lemma 5].

If a non-associative algebra \mathfrak{A} is homomorphic to a semi-simple algebra (direct sum of simple algebras), there is an ideal \mathfrak{R} of \mathfrak{A} , called the *radical* of \mathfrak{A} , such that $\mathfrak{A} - \mathfrak{R}$ is semi-simple and \mathfrak{R} is contained in every ideal \mathfrak{B} of \mathfrak{A} such that $\mathfrak{A} - \mathfrak{B}$ is semi-simple. It is an immediate consequence of [2, Theorem 6] that any nilpotent ideal of \mathfrak{A} is contained in the radical of \mathfrak{A} .

2. Baric algebras. A non-associative algebra \mathfrak{A} of order n over a field \mathfrak{F} is called *baric* in case it has a non-trivial representation of degree one—that is, in case there is a homomorphism (3) of \mathfrak{A} into \mathfrak{F} such that for some x_0 in \mathfrak{A} we have $\omega(x_0) \neq 0$. It follows that ω is a homomorphism of \mathfrak{A} onto \mathfrak{F} since, for any α in \mathfrak{F} , we have $\omega(\alpha x_0 / \omega(x_0)) = \alpha$. We call $\omega(x)$ the *weight* of x , and ω the *weight function* of \mathfrak{A} .

We denote the kernel of the homomorphism ω by \mathfrak{R} . Then a necessary and sufficient condition that a non-associative algebra \mathfrak{A} be a baric algebra is that \mathfrak{A} contain an ideal \mathfrak{R} such that

$$(5) \quad \mathfrak{A} - \mathfrak{R} \cong \mathfrak{F}.$$

Thus any non-associative algebra \mathfrak{R} of order $n - 1$ over \mathfrak{F} gives rise to a baric algebra \mathfrak{A} of order n over \mathfrak{F} if we adjoin an element u to \mathfrak{R} in any fashion such that the elements $u^2 - u$, uz , and zu are in \mathfrak{R} for all z in \mathfrak{R} (a trivial construction).

In a *gametic algebra* \mathfrak{A} we take a basis u_1, u_2, \dots, u_n denoting the gametic types involved in some genetical situation of symmetrical inheritance [5, § 6]. If γ_{ijk} is the probability that an arbitrary gamete produced by an individual of zygotic type $u_i u_j$ ($= u_j u_i$) be of gametic type u_k , we have

$$(6) \quad u_i u_j = \sum_k \gamma_{ijk} u_k \quad (i, j = 1, \dots, n)$$

subject to the conditions

$$(7) \quad \sum_k \gamma_{ijk} = 1 \quad (i, j = 1, \dots, n).$$

Etherington points out that equations (6) and (7)—not assuming commutativity—imply that \mathfrak{A} is a baric algebra with weight function

$$(8) \quad \omega: x = \sum_i \xi_i u_i \rightarrow \omega(x) = \sum_i \xi_i, \quad \xi_i \text{ in } \mathfrak{F}.$$

The converse is also true: given any baric algebra \mathfrak{A} there is a basis u_1, u_2, \dots, u_n of \mathfrak{A} whose multiplication table (6) is subject to the conditions (7). Also the defining homomorphism ω has the form (8). For let v_2, \dots, v_n be a basis of the ideal \mathfrak{N} of \mathfrak{A} . There exists an element u in \mathfrak{A} , but not in \mathfrak{N} , of weight 1, so that \mathfrak{A} has basis u, v_2, \dots, v_n . Write $u_1 = u, u_i = u + v_i$ ($i = 2, \dots, n$). Then

$$(9) \quad \omega(u_i) = 1 \quad (i = 1, \dots, n).$$

Moreover, \mathfrak{A} has the basis u_1, u_2, \dots, u_n satisfying (6) for some γ_{ijk} in \mathfrak{F} . Now (4) and (9) imply that $1 = \omega(u_i)\omega(u_j) = \omega(u_i u_j) = \sum_k \gamma_{ijk} \omega(u_k) = \sum_k \gamma_{ijk}$ so that (7) holds. Also (8) follows from (9).

If \mathfrak{A} is a baric algebra, then $T(\mathfrak{A})$ is also a baric algebra. For if \mathfrak{A} has weight function ω , a weight function θ for $T(\mathfrak{A})$ is defined by

$$(10) \quad \theta(T) = \omega(uT),$$

where u is any element of weight 1 in \mathfrak{A} . That θ is well-defined by (10) is clear since, if T in $T(\mathfrak{A})$ is written in the form (1), we have equivalently

$$(11) \quad \theta(T) = \alpha + f(\omega(x_1), \omega(x_1), \omega(x_2), \dots).$$

Then θ is linear by (10), a homomorphism by (11), and non-trivial since $\theta(I) = 1$.

3. Genetic algebras. Etherington has investigated the non-commutative aspects of some of the concepts he has introduced. However, since any algebra encountered in genetics may be taken to be commutative [8, p. 26], we shall assume the commutative law in all that follows.

In a commutative algebra \mathfrak{A} we have $L_x = R_x$ for all x , so that we may write T in $T(\mathfrak{A})$ in the form

$$(12) \quad T = \alpha I + f(R_{x_1}, R_{x_2}, \dots), \quad \alpha \text{ in } \mathfrak{F}, x_i \text{ in } \mathfrak{A}.$$

The characteristic function $|\lambda I - T|$ of T in (12) has coefficients which are polynomials in α and the coordinates of the x_i , polynomials which depend

both on the function f and on elements of \mathfrak{F} which are independent of T (that is, scalars completely determined by \mathfrak{A}).

We call a commutative baric algebra \mathfrak{A} over \mathfrak{F} with weight function ω a *genetic algebra* in case the coefficients of the characteristic function of T in (12), insofar as they depend on the x_i , depend only on the weights $\omega(x_i)$. That is, these coefficients are polynomials in α and the $\omega(x_i)$ having coefficients which involve certain elements of \mathfrak{F} determined by \mathfrak{A} in combinations determined by f . (Note: the fact that for a given T in $T(\mathfrak{A})$ the expression (12) is not unique has no bearing on our definition.)

This definition is an extension of Etherington's definition of train algebra [5, § 4]. Define the *right powers* x^k of x in \mathfrak{A} by $x^1 = x$ and

$$(13) \quad x^k = xR_x^{k-1} \quad (k = 2, 3, \dots).$$

(Since we assume \mathfrak{A} commutative, right powers and similarly defined *left powers* are equal.) Then a commutative baric algebra \mathfrak{A} over \mathfrak{F} with weight function ω is called a *train algebra* in case the coefficients of the (right) rank equation [4, § 19], insofar as they depend on x , depend only on $\omega(x)$. That is, there exist elements $\beta_1, \dots, \beta_{r-1}$ in \mathfrak{F} such that

$$(14) \quad x^r + \beta_1\omega(x)x^{r-1} + \dots + \beta_{r-1}[\omega(x)]^{r-1}x = 0$$

for all x in \mathfrak{A} , where x^k is the right power (13).

THEOREM 1. *A genetic algebra \mathfrak{A} over \mathfrak{F} is a train algebra.*

Let $T = R_x$ in (12), and write $\omega(x) = \xi$. Then, since the coefficients of the characteristic function

$$\phi(\lambda) = |\lambda I - R_x|$$

of R_x are homogeneous polynomials in the coordinates of x , we have, by the definition of a genetic algebra,

$$(15) \quad \phi(\lambda) = \lambda^n + \gamma_1\xi\lambda^{n-1} + \dots + \gamma_n\xi^n,$$

for some $\gamma_1, \dots, \gamma_n$ in \mathfrak{F} . Now (15) factors in a finite extension \mathfrak{R} of \mathfrak{F} as

$$(16) \quad \phi(\lambda) = (\lambda - \lambda_1\xi)(\lambda - \lambda_2\xi) \dots (\lambda - \lambda_n\xi), \quad \lambda_i \text{ in } \mathfrak{R}.$$

If

$$(17) \quad \lambda^r + \psi_1\lambda^{r-1} + \dots + \psi_{r-1}\lambda$$

is the rank function of \mathfrak{A} , ψ_j a homogeneous polynomial of degree j in the coordinates of x , then (17) divides $\lambda\phi(\lambda)$ [4, § 19]. The λ_i in (16) may

then be ordered so that (17) equals $\lambda(\lambda - \lambda_1\xi) \cdots (\lambda - \lambda_{r-1}\xi)$, from which it follows that

$$\psi_j = (-1)^{j\xi^j \sum \lambda_{i_1} \cdots \lambda_{i_j}}, \quad j = 1, 2, \dots, r - 1.$$

The rank equation is (14) with

$$\beta_j = (-1)^{j \sum \lambda_{i_1} \cdots \lambda_{i_j}}, \quad j = 1, 2, \dots, r - 1,$$

in \mathfrak{F} ; \mathfrak{A} is a train algebra.

In a non-associative algebra in which powers of a single element are not necessarily associative, the concept of a nilpotent element may be defined variously. Here we shall call an element z *nilpotent* in case there exists an integer k for which the right power $z^k = 0$. In a train algebra \mathfrak{A} the kernel \mathfrak{N} of the weight function ω then has an easy characterization: \mathfrak{N} consists of the nilpotent element of \mathfrak{A} . For $z^k = 0$ implies $\omega(z^k) = [\omega(z)]^k = 0$, $\omega(z) = 0$; conversely $\omega(z) = 0$ implies $z^r = 0$ by (14). It follows from Theorem 1 that the same characterization of \mathfrak{N} holds for genetic algebras.

We construct an example of a train algebra which is not a genetic algebra as follows: let \mathfrak{F} have characteristic two.¹ Then the square of any element z in the commutative algebra $\mathfrak{N} = (v_1, v_2, v_3)$ with multiplication table

$$v_1v_2 = v_3, \quad v_2v_3 = v_1, \quad v_3v_1 = v_2, \quad v_i^2 = 0 \quad (i = 1, 2, 3)$$

is zero. However, \mathfrak{N} is not a nilpotent algebra, since $\mathfrak{N} = \mathfrak{N}^2 (= \mathfrak{N}\mathfrak{N})$. Let \mathfrak{A} be the algebra obtained by adjoining a unity element 1 to \mathfrak{N} . Then x in \mathfrak{A} has the form $x = \xi 1 + z$, and \mathfrak{A} is a train algebra since $x \rightarrow \xi$ is a weight function for \mathfrak{A} , while $z^2 = (x - \xi 1)^2 = x^2 + \xi^2 1 = 0$ implies $x^3 + \xi^2 x = 0$. Since \mathfrak{N} is not nilpotent, it follows from Theorem 4 below that \mathfrak{A} is not a genetic algebra. This example also shows that a structure theory as elementary as that in **5** below is not possible for train algebras.

A commutative baric algebra \mathfrak{A} with weight function ω is called a *special train algebra* in case

- (a) the kernel \mathfrak{N} of ω is nilpotent, and
- (b) the subalgebras \mathfrak{N}^k of \mathfrak{A} defined inductively by $\mathfrak{N}^1 = \mathfrak{N}$, $\mathfrak{N}^k = \mathfrak{N}^{k-1}\mathfrak{N}$ for $k = 2, 3, \dots$, are ideals of \mathfrak{A} .

¹ If we knew an example, over a more or less arbitrary field, of a commutative non-nilpotent algebra \mathfrak{N} , all of whose elements are nilpotent, we could give a more satisfying example of a train algebra which is not a genetic algebra by adjoining 1 to \mathfrak{N} . There are many examples in the literature of non-commutative algebras \mathfrak{N} with these properties but, although it seems possible that commutative examples exist, we have not been able to construct one.

THEOREM 2. *A special train algebra \mathfrak{A} over \mathfrak{F} is a genetic algebra.*

Etherington has shown in [7, § 4] that over a finite extension \mathfrak{R} of \mathfrak{F} there exists a basis of $\mathfrak{A}_{\mathfrak{R}}$, together with scalars $\lambda_1 = 1, \lambda_2, \dots, \lambda_n$ in \mathfrak{R} , such that the matrix of R_x for x in \mathfrak{A} has the form

$$(18) \quad R_x = \begin{pmatrix} \xi\lambda_1 & * & \dots & * \\ 0 & \xi\lambda_2 & \dots & * \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \xi\lambda_n \end{pmatrix}, \quad \xi = \omega(x).$$

Then the characteristic function (15) of R_x has the form (16) with λ_i as in (18). Hence

$$(19) \quad (-1)^t \gamma_t = \Sigma \lambda_{i_1} \dots \lambda_{i_t} \quad (t = 1, 2, \dots, n),$$

and the γ_t in \mathfrak{F} are dependent, not on x , but only on the algebra \mathfrak{A} . From (18) we obtain

$$f(R_{x_1}, R_{x_2}, \dots) = \begin{pmatrix} f(\xi_1\lambda_1, \xi_2\lambda_1, \dots) & * & \dots & * \\ 0 & f(\xi_1\lambda_2, \xi_2\lambda_2, \dots) & \dots & * \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & f(\xi_1\lambda_n, \xi_2\lambda_n, \dots) \end{pmatrix},$$

where $\xi_k = \omega(x_k)$. Then T in (12) has characteristic equation

$$\begin{aligned} |\lambda I - T| &= |(\lambda - \alpha)I - f(R_{x_1}, R_{x_2}, \dots)| \\ &= [(\lambda - \alpha) - \eta_1][(\lambda - \alpha) - \eta_2] \dots [(\lambda - \alpha) - \eta_n] = 0 \end{aligned}$$

where we have written

$$(20) \quad \eta_i = f(\xi_1\lambda_i, \xi_2\lambda_i, \dots) \quad (n = 1, 2, \dots, n).$$

Then

$$|\lambda I - T| = (\lambda - \alpha)^n + \mu_1(\lambda - \alpha)^{n-1} + \dots + \mu_n,$$

where $(-1)^s \mu_s$ is the elementary symmetric function of degree s in the η_i . But then by (20) the μ_s ($s = 1, 2, \dots, n$) are polynomials in ξ_1, ξ_2, \dots , with coefficients which are symmetric functions of the λ_i . These coefficients are expressible in terms of the elementary symmetric functions (19) of the λ_i , and therefore are not dependent on the x_i . Hence \mathfrak{A} is a genetic algebra.

The copular algebra of simple mendelian inheritance (6 below) is an example of a genetic algebra which is not a special train algebra.

4. Duplicate of a commutative algebra. Let \mathfrak{A} be a commutative algebra of order n over \mathfrak{F} , and let u_1, u_2, \dots, u_n be a basis of \mathfrak{A} with multiplication table (6). The *duplicate* \mathfrak{A}' of \mathfrak{A} is defined as the commutative algebra of order $\frac{1}{2}n(n+1)$ over \mathfrak{F} with basal elements v_{ij} ($i \leqq j$; $i, j = 1, 2, \dots, n$) satisfying

$$(21) \quad v_{ij}v_{rs} = \sum_{kt} \gamma_{ijk} \gamma_{rst} v_{kt} \quad (i \leqq j, r \leqq s; i, j, r, s, k, t = 1, \dots, n)$$

where we identify $v_{ik} = v_{kt}$ for $t > k$. That is, element of \mathfrak{A}' behave like quadratic forms in \mathfrak{A} . The definition of \mathfrak{A}' is independent of the basis chosen for \mathfrak{A} since $\mathfrak{A} \cong \mathfrak{A}_1$ implies $\mathfrak{A}' \cong \mathfrak{A}'_1$ [9, Theorem IV].

The process of duplication is important in genetics because we obtain from any gametic algebra \mathfrak{A} a corresponding *zygotic algebra* \mathfrak{A}' whose basis consists of the zygotic types $u_i u_j$ ($= u_j u_i$) obtained from the gametic types u_1, u_2, \dots, u_n in \mathfrak{A} . Multiplication in the zygotic algebra \mathfrak{A}' is carried out as though it were being performed in \mathfrak{A} according to the multiplication table (6). Writing v_{ij} for $u_i u_j$ ($i \leqq j$) we obtain (21), where the coefficient of v_{kt} is the probability that an individual of zygotic type $u_i u_j$ mating with one of type $u_r u_s$ will produce an individual of zygotic type $u_k u_t$. If ω is the weight function (8) of the gametic algebra \mathfrak{A} , then

$$\omega': a = \sum_{ij} \alpha_{ij} v_{ij} \rightarrow \omega'(a) = \sum_{ij} \alpha_{ij}, \quad \alpha_{ij} \text{ in } \mathfrak{F},$$

is a weight function for the zygotic algebra \mathfrak{A}' ; \mathfrak{A}' is a baric algebra. Genetical calculations involving the first filial generation may be performed in \mathfrak{A}' , those involving the second filial generation in the *copular algebra* \mathfrak{A}'' (the duplicate of \mathfrak{A}'), etc.

We return to the general notion of a duplicate algebra as defined by (21). There is a homomorphism H of \mathfrak{A}' into \mathfrak{A} defined by

$$(22) \quad H: v_{ij} \rightarrow v_{ij}H = u_i u_j = \sum_k \gamma_{ijk} u_k, \quad (i \leqq j; i, j = 1, \dots, n).$$

Actually H is a homomorphism of \mathfrak{A}' onto the ideal \mathfrak{A}^2 ($= \mathfrak{A}\mathfrak{A}$) of \mathfrak{A} , since the $u_i u_j$ span \mathfrak{A}^2 . We denote the kernel of H by \mathfrak{D} . It is easy to see that

$$(23) \quad \mathfrak{D}\mathfrak{A}' = 0$$

[9, Theorem II (ii)]; that is, \mathfrak{D} consists of absolute divisors of zero.

Let a be an element of \mathfrak{A}' . We denote the corresponding right multiplication of \mathfrak{A}' by R^*_a . Then any element T_* of $T(\mathfrak{A}')$ has the form

$$(24) \quad T_* = \alpha I_* + f(R^*_{a_1}, R^*_{a_2}, \dots), \quad \alpha \text{ in } \mathfrak{F}, a_i \text{ in } \mathfrak{A}',$$

where I_* is the identity on \mathfrak{A}' .

LEMMA. Let a commutative algebra \mathfrak{A} of order n over \mathfrak{F} have duplicate \mathfrak{A}' , and T_* in $T(\mathfrak{A}')$ have the form (24). Then the characteristic function of T_* is

$$(25) \quad |\lambda I_* - T_*| = (\lambda - \alpha)^{\frac{1}{2}n(n-1)} |\lambda I - T|$$

where T in $T(\mathfrak{A})$ has the form (12) with $x_i = a_i H$, and H is the homomorphism (22) of \mathfrak{A}' into \mathfrak{A} .

Let m be the order over \mathfrak{F} of the kernel \mathfrak{D} of H . Then $\mathfrak{A}' = \mathfrak{D} + \mathfrak{D}$ for a linear subspace \mathfrak{D} of \mathfrak{A}' having order $p = \frac{1}{2}n(n+1) - m$ over \mathfrak{F} . It follows from (23) that, corresponding to this way of writing \mathfrak{A}' , the right multiplication R^*_a has matrix

$$R^*_a = \begin{pmatrix} 0 & 0 \\ M_a & N_a \end{pmatrix}, \quad a \text{ in } \mathfrak{A}'$$

where M_a and N_a are $p \times m$ and $p \times p$ matrices respectively. Since $\mathfrak{A}' - \mathfrak{D} \cong \mathfrak{A}^2$ under the natural correspondence determined by H , we have $N_a = R^0_{aH}$, the right multiplication of \mathfrak{A}^2 corresponding to aH in \mathfrak{A}^2 . Then R^*_a has matrix

$$(26) \quad R^*_a = \begin{pmatrix} 0 & 0 \\ * & R^0_x \end{pmatrix}, \quad x = aH \text{ in } \mathfrak{A}^2.$$

Now $\mathfrak{A} = \mathfrak{A}^2 + \mathfrak{G}$ for a linear subspace \mathfrak{G} of \mathfrak{A} having order $n - p$ over \mathfrak{F} . Corresponding to this way of writing \mathfrak{A} , the matrix of R_x for x in $\mathfrak{A}^2 \subset \mathfrak{A}$ is

$$(27) \quad R_x = \begin{pmatrix} R^0_x & 0 \\ * & 0 \end{pmatrix}, \quad x \text{ in } \mathfrak{A}^2,$$

since \mathfrak{A}^2 is an ideal of \mathfrak{A} . It follows from (26) and (27) that

$$(28) \quad f(R^*_{a_1}, R^*_{a_2}, \dots) = \begin{pmatrix} 0 & 0 \\ * & f(R^0_{x_1}, R^0_{x_2}, \dots) \end{pmatrix}, \quad x_i = a_i H,$$

and

$$(29) \quad f(R_{x_1}, R_{x_2}, \dots) = \begin{pmatrix} f(R^0_{x_1}, R^0_{x_2}, \dots) & 0 \\ * & 0 \end{pmatrix}, \quad x_i \text{ in } \mathfrak{A}^2.$$

Then, denoting by I_p the (p -rowed) identity on \mathfrak{A}^2 and \mathfrak{D} , equations (28) and (29) imply that $|\lambda I_* - T_*| = (\lambda - \alpha)^m |(\lambda - \alpha)I_p - f(R^0_{x_1}, R^0_{x_2}, \dots)| = (\lambda - \alpha)^{m-(n-p)} |\lambda I - T|$, for T_* in (24) and T in (12) with $x_i = a_i H$. Equation (25) follows immediately since $m - n + p = -n + \frac{1}{2}n(n+1) = \frac{1}{2}n(n-1)$.

We use this lemma in the proof of

THEOREM 3. *The duplicate \mathfrak{A}' of a genetic algebra \mathfrak{A} over \mathfrak{F} is itself a genetic algebra.*

By definition \mathfrak{A}' is commutative. If ω is the weight function of \mathfrak{A} , then a weight function ω' of \mathfrak{A}' is defined by

$$(30) \quad \omega': a \rightarrow \omega'(a) = \omega(aH), \quad a \text{ in } \mathfrak{A}', aH \text{ in } \mathfrak{A},$$

where H is the homomorphism (22) of \mathfrak{A}' into \mathfrak{A} . It follows from (25) and the fact that \mathfrak{A} is a genetic algebra that the characteristic function of T^* in (24) has coefficients which, insofar as they depend on the a_i , depend only on the $\omega(x_i) = \omega(a_iH) = \omega'(a_i)$; \mathfrak{A}' is a genetic algebra.

The advantage of the concept of genetic algebra over special train algebra lies in Theorem 3. The most elementary algebra of genetics, the *gametic algebra of simple mendelian inheritance* (6 below), is a special train algebra (therefore a genetic algebra by Theorem 2). Hence by Theorem 3 all algebras obtained from it by duplication are also genetic algebras. However, the copular algebra of simple mendelian inheritance, obtained by duplicating the gametic algebra twice, is not a special train algebra.

5. Structure of genetic algebras. It follows from (5) that the radical of any baric algebra is contained in the kernel \mathfrak{R} of the weight function ω . We shall show that for a genetic algebra \mathfrak{A} the radical is \mathfrak{R} , by showing that \mathfrak{R} , which we already know consists of the nilpotent elements of \mathfrak{A} , is actually nilpotent.

THEOREM 4. *Let \mathfrak{R} be the kernel of the weight function ω of a genetic algebra \mathfrak{A} over \mathfrak{F} . Then \mathfrak{R} is the radical of \mathfrak{A} , and is nilpotent.*

Let T in $T(\mathfrak{A})$ have the form (12), and write $\omega(x_i) = \xi_i$. Then the characteristic function

$$|\lambda I - T| = \lambda^n + \psi_1 \lambda^{n-1} + \dots + \psi_n$$

of T has coefficients ψ_j which are polynomials in $\alpha, \xi_1, \xi_2, \dots$, with constant terms $\psi_{j0} = 0$. For let $\alpha = 0$ and $x_i = 0$ ($i = 1, 2, \dots$) in (12); then $T = 0$, $|\lambda I| = \lambda^n = \lambda^n + \psi_{10} \lambda^{n-1} + \dots + \psi_{n0}$, or $\psi_{j0} = 0$ for $j = 1, 2, \dots, n$. It follows that λ^n is the characteristic function of any T in $T(\mathfrak{A})$ which may be written in the form (12) with

$$(31) \quad \alpha = \omega(x_i) = 0, \quad i = 1, 2, \dots$$

In this case $T^n = 0$, T is nilpotent. Now let T be in the enveloping algebra

\mathfrak{N}^* of the right multiplications corresponding to elements of \mathfrak{N} . Then (31) is satisfied, T is nilpotent. Since \mathfrak{N}^* is an associative algebra consisting of nilpotent elements, \mathfrak{N}^* is nilpotent. Thus \mathfrak{N} is a nilpotent ideal of \mathfrak{A} , and is contained in the radical \mathfrak{R} of \mathfrak{A} . On the other hand, (5) implies that \mathfrak{N} contains \mathfrak{R} , or $\mathfrak{N} = \mathfrak{R}$.

The classical elements of a structure theory for a linear algebra \mathfrak{A} are

- (i) the nature of the radical \mathfrak{R} , and
- (ii) the nature of the simple components of the semi-simple algebra $\mathfrak{A} - \mathfrak{R}$.

For genetic algebras, (i) is answered by Theorem 4. Question (ii) is trivial by virtue of (5).

One may also ask whether or not the analogue of the so-called Wedderburn Principal Theorem holds: does \mathfrak{A} contain a subalgebra $\mathfrak{S} \cong \mathfrak{A} - \mathfrak{R}$, so that $\mathfrak{A} = \mathfrak{S} + \mathfrak{R}$? It is easy to see from (5) that this question is equivalent to the following one: does \mathfrak{A} contain an idempotent element e ? One may readily construct an example of a genetic algebra without an idempotent, so the answer in general is negative.

However, the existence of an idempotent is significant genetically, since it represents a population in equilibrium for random mating [6, p. 138]. Etherington gives conditions for the existence of an idempotent in a commutative baric algebra [6, Theorem VI]. Clearly a genetic algebra (or even a train algebra) contains an idempotent e if and only if there is an associative subalgebra \mathfrak{C} of \mathfrak{A} which is not contained in \mathfrak{N} . For e generates such an algebra \mathfrak{C} , while the converse follows from the fact that any non-nilpotent associative algebra \mathfrak{C} contains an idempotent.

6. Simple mendelian inheritance. Let \mathfrak{F} have characteristic not two. The *gametic algebra* \mathfrak{G} of *simple mendelian inheritance* is the commutative algebra $\mathfrak{G} = (u_1, u_2)$ of order 2 over \mathfrak{F} with gametic multiplication table

$$(32) \quad u_1^2 = u_1, \quad u_1u_2 = \frac{1}{2}u_1 + \frac{1}{2}u_2, \quad u_2^2 = u_2.$$

An easy change of basis gives $\mathfrak{G} = (u, z)$ with

$$(33) \quad u^2 = u, \quad uz = \frac{1}{2}z, \quad z^2 = 0.$$

Writing $x = \xi u + \eta z$, we have weight function $\omega : x \rightarrow \omega(x) = \xi$; then $\mathfrak{N} = (z)$, $\mathfrak{N}^2 = 0$, \mathfrak{G} is a special train algebra. The transformation algebra $T(\mathfrak{G})$ has order 3 over \mathfrak{F} , and any element T of $T(\mathfrak{G})$ may be written in the form

$$(34) \quad T = \alpha I + \beta R_x, \quad \alpha \text{ in } \mathfrak{F}, x = \xi u + \eta z \text{ in } \mathfrak{G}.$$

The characteristic function of T in (34) is

$$(35) \quad \lambda^2 - (2\alpha + 3\xi)\lambda + (\alpha + \xi)(\alpha + 2\xi) = [\lambda - (\alpha + 2\xi)][\lambda - (\alpha + \xi)].$$

The *zygotic algebra* \mathfrak{Z} of *simple mendelian inheritance* is obtained from \mathfrak{G} by duplication. For purposes of computation in genetics one would want the multiplication table obtained by duplicating (32). However, the structure of $\mathfrak{Z} = \mathfrak{G}$ is seen more easily when the multiplication table (33) is duplicated. Write $a = uu$, $b = uz$, $c = zz$. Then $\mathfrak{Z} = (a, b, c)$ with

$$(36) \quad a^2 = a, \quad ab = \frac{1}{2}b, \quad b^2 = \frac{1}{4}c, \quad ac = bc = c^2 = 0.$$

Writing $x = \xi a + \eta b + \zeta c$, we have the weight function $\omega: x \rightarrow \omega(x) = \xi$. The kernel of ω is $\mathfrak{N} = (b, c)$. Then $\mathfrak{N}^2 = (c)$, $\mathfrak{N}^3 = 0$, \mathfrak{Z} is a special train algebra. The transformation algebra $T(\mathfrak{Z})$ has order 6 over \mathfrak{F} and any element T of $T(\mathfrak{Z})$ may be written in the form

$$(37) \quad T = \alpha I + 2R_{x_1} + 4R_{x_2}R_{x_3}, \quad x_i = \xi_i a + \eta_i b, \quad \omega(x_i) = \xi_i.$$

The characteristic function of T in (37) is

$$(38) \quad [\lambda - \alpha][\lambda - (\alpha + 2\xi_1 + 4\xi_2\xi_3)][\lambda - (\alpha + \xi_1 + \xi_2\xi_3)].$$

Duplication of \mathfrak{Z} gives the *copular algebra* \mathfrak{C} of *simple mendelian inheritance*. As before we omit the multiplication table (important from the point of view of genetics, but not structurally) which is obtained by duplicating (32) twice. Instead we write $v = aa$, $p_1 = ab$, $p_2 = bb$, $p_3 = ac$, $p_4 = bc$, $p_5 = cc$, for a, b, c in (36). Then $\mathfrak{C} = (v, p_1, \dots, p_5)$ with

$$(39) \quad \begin{aligned} v^2 &= v, & vp_1 &= \frac{1}{2}p_1, & vp_2 &= \frac{1}{4}p_3, & p_1^2 &= \frac{1}{4}p_2, & p_1p_2 &= \frac{1}{8}p_4, \\ p_2^2 &= \frac{1}{16}p_5, & vp_j &= p_ip_j = 0 & & & & & & (i = 1, \dots, 5; j = 3, 4, 5). \end{aligned}$$

Writing $x = \xi v + \sum \pi_i p_i$, we have $\omega(x) = \xi$. The kernel of ω is $\mathfrak{N} = (p_1, \dots, p_5)$, and $\mathfrak{N}^2 = (p_2, p_4, p_5)$, $\mathfrak{N}^3 = (p_4, p_5)$, $\mathfrak{N}^4 = 0$. Now \mathfrak{C} is not a special train algebra since $\mathfrak{C}\mathfrak{N}^2$ contains $vp_2 = \frac{1}{4}p_3$, which is not in \mathfrak{N}^2 ; \mathfrak{N}^2 is not an ideal of \mathfrak{C} . That \mathfrak{C} is a genetic algebra is guaranteed by Theorem 3.

7. Jordan algebras. A commutative algebra \mathfrak{A} of order n over \mathfrak{F} is called a *Jordan algebra* in case

$$x^2(xy) = x(x^2y) \quad \text{for all } x, y \text{ in } \mathfrak{A}.$$

Let \mathfrak{F} have characteristic not two; any linear subspace \mathfrak{M} of $(\mathfrak{F})_m$ which is closed with respect to “quasi-multiplication”

$$(40) \quad xy = \frac{1}{2}(x \cdot y + y \cdot x), \quad x, y \text{ in } \mathfrak{M},$$

where $x \cdot y$ denotes the associative multiplication of transformations in \mathfrak{M} , is a *Jordan algebra of linear transformations* of order $n \leq m^2$ over \mathfrak{F} [3, p. 546].

The gametic algebra \mathfrak{G} of simple mendelian inheritance is the case $n = 2$ of the genetic algebra $\mathfrak{G}_n = (u, z_2, \dots, z_n)$ with multiplication table

$$(41) \quad u^2 = u, \quad uz_j = \frac{1}{2}z_j, \quad z_i z_j = 0 \quad (i, j = 2, \dots, n).$$

In [6, § 6] Etherington has shown that any train algebra of rank 2 and order n over \mathfrak{F} of characteristic not two is equivalent to \mathfrak{G}_n , and also has indicated that these algebras are Jordan algebras [6, p. 138, footnote]. Actually they are Jordan algebras of linear transformations. For let e_{ij} ($i, j = 1, \dots, n$) be the usual matrix basis of $(\mathfrak{F})_n$ with matrix multiplication

$$(42) \quad e_{ij} \cdot e_{kl} = \delta_{jk} e_{il} \quad (\text{Kronecker delta}),$$

and let $\mathfrak{M} = (e_{11}, e_{12}, \dots, e_{1n})$. By (40) \mathfrak{M} is a Jordan algebra of linear transformations with multiplication $e_{1i} e_{1j} = \frac{1}{2}(\delta_{i1} e_{1j} + \delta_{j1} e_{1i})$ or

$$(43) \quad e_{11}^2 = e_{11}, \quad e_{11} e_{1j} = \frac{1}{2} e_{1j}, \quad e_{1i} e_{1j} = 0 \quad (i, j = 2, \dots, n).$$

By (41) and (43) the correspondence $u \rightarrow e_{11}, z_i \rightarrow e_{1i}$ ($i = 2, \dots, n$) is an equivalence between \mathfrak{G}_n and \mathfrak{M} .

The zygotic algebra \mathfrak{Z} of simple mendelian inheritance is also a Jordan algebra of linear transformations. For let \mathfrak{M} be the subspace of $(\mathfrak{F})_3$ with basal elements $a = e_{22}, b = e_{12} + e_{23}, c = 4e_{13}$. Defining multiplication in \mathfrak{M} by (40) and (42), we obtain the multiplication table (36); \mathfrak{M} is a Jordan algebra of linear transformations and is equivalent to \mathfrak{Z} .

Inasmuch as powers of a single element are associative in a Jordan algebra over \mathfrak{F} of characteristic not two [3, § 5], the copular algebra \mathfrak{C} of simple mendelian inheritance is not a Jordan algebra. For we see from (39) that the right power $p_1^4 = 0$, while $p_1^2 p_1^2 = 2^{-8} p_5 \neq 0$.

These considerations lead us to an analysis of those genetic algebras \mathfrak{A} over a field \mathfrak{F} of characteristic not two which are at the same time Jordan algebras. Let u be an element of weight 1 in \mathfrak{A} . Then, by the associativity of powers in \mathfrak{A} , u generates an associative subalgebra \mathfrak{C} which is not contained in \mathfrak{A} ; there is an idempotent e in \mathfrak{A} .

Albert has shown in [3] that the only possible characteristic roots of R_e are 0, $\frac{1}{2}$, 1, so that the equation

$$(44) \quad xe = \beta x, \quad \beta \text{ in } \mathfrak{F},$$

has solutions only for $\beta = 0, \frac{1}{2}, 1$. Writing $\mathfrak{A}_e(\beta)$ for the set of all x in \mathfrak{A} satisfying (44), one obtains \mathfrak{A} as the supplementary sum

$$(45) \quad \mathfrak{A} = \mathfrak{A}_e(1) + \mathfrak{A}_e(\frac{1}{2}) + \mathfrak{A}_e(0).$$

Let t_β be the dimension of the space $\mathfrak{A}_e(\beta)$ over \mathfrak{F} . Then $t_1 \geq 1$ and $t_1 + t_{\frac{1}{2}} + t_0 = n$. Corresponding to a basis of \mathfrak{A} in the form (45), we have the matrix of R_e in the diagonal form

$$(46) \quad R_e = \text{diag}\{I_1, \frac{1}{2}I_{\frac{1}{2}}, 0\}$$

where I_β is the t_β -rowed identity matrix.

Since $T(\mathfrak{A})$ is a baric algebra with weight function θ defined by (10), or

$$(47) \quad \theta(T) = \alpha + f(\xi_1, \xi_2, \dots), \quad \xi_i = \omega(x_i),$$

for T in (12), we know that one characteristic root of T is $\theta(T)$. For $\phi(\lambda) = |\lambda I - T|$ implies that $\phi(T) = 0$ and $\theta[\phi(T)] = \phi[\theta(T)] = 0$, $\theta(T)$ is a root of $\phi(\lambda) = 0$. However, for Jordan algebras we can obtain the following stronger result.

THEOREM 5. *Let \mathfrak{F} have characteristic not two, and \mathfrak{A} be a genetic algebra which is a Jordan algebra over \mathfrak{F} . Then \mathfrak{A} contains an idempotent e , and the distinct characteristic roots of T in $T(\mathfrak{A})$ in (12) are at most three:*

$$\alpha, \quad \theta(T), \quad \alpha + f(\frac{1}{2}\xi_1, \frac{1}{2}\xi_2, \dots),$$

where $\xi_i = \omega(x_i)$, and θ is the weight function (10) of $T(\mathfrak{A})$. The multiplicities of these roots are the orders over \mathfrak{F} of $\mathfrak{A}_e(0)$, $\mathfrak{A}_e(1)$, $\mathfrak{A}_e(\frac{1}{2})$ respectively.

For it follows from (46) that

$$(48) \quad f(R_{\xi_1 e}, R_{\xi_2 e}, \dots) = \text{diag}\{f(\xi_1, \xi_2, \dots)I_1, f(\frac{1}{2}\xi_1, \frac{1}{2}\xi_2, \dots)I_{\frac{1}{2}}, 0\}.$$

Then, since \mathfrak{A} is a genetic algebra and $\omega(x_i) = \xi_i = \omega(\xi_i e)$, we have

$$|\lambda I - T| = |(\lambda - \alpha)I - f(R_{x_1}, R_{x_2}, \dots)| = |(\lambda - \alpha)I - f(R_{\xi_1 e}, R_{\xi_2 e}, \dots)|.$$

Then (48) and (47) imply that the characteristic function of T is

$$(49) \quad [\lambda - \alpha]^{t_0} [\lambda - \theta(T)]^{t_1} [\lambda - \{\alpha + f(\frac{1}{2}\xi_1, \frac{1}{2}\xi_2, \dots)\}]^{t_{\frac{1}{2}}}.$$

The theorem follows.

In the examples we have worked out for simple mendelian inheritance, (35) and (38) illustrate the characteristic function (49).

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